Chain Rule

We know, $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(x^3) = 3x^2$

So can we say anything on $\frac{d}{dx}\left(\sin(x^3)\right)$ at $x=a$?

Let's use the definition of derivative: $f(x) = \sin(x^3)$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\sin(x^3)-\sin(a^3)}{x-a}$$

$$= \lim_{x\to a}\left[\frac{\sin(x^3)-\sin(a^3)}{x^3-a^3} \cdot \frac{x^3-a^3}{x-a}\right]$$

$$= \boxed{\lim_{x\to a} \frac{\sin(x^3)-\sin(a^3)}{x^3-a^3}} \cdot \boxed{\lim_{x\to a} \frac{x^3-a^3}{x-a}}$$

$[x^3]'$ at $x=a$

$= 3a^2$

Let, $u = x^3$

Then $x \to a \Rightarrow u \to a^3$

$$= \boxed{\lim_{u\to a^3} \frac{\sin(u)-\sin(a^3)}{u-a^3}}$$

$[\sin u]'_{u=a^3}$

$= \cos u \Big|_{u=a^3}$

$= \cos(a^3)$

So, $\frac{d}{dx}\left[\sin(x^3)\right]\Big|_{x=a} = \cos(a^3) \cdot 3a^2$.

Note: If we write $g(x) = x^3$ & $h(x) = \sin x$

then $\sin(x^3) = \sin(g(x)) = h(g(x)) = \underbrace{(h \circ g)}_{\text{composition}}(x)$

Let $f(x)$, $g(x)$ be functions & let $h(x) = (f \circ g)(x)$
$= f(g(x))$.

Then $h'(x) = \underbrace{f'(g(x))}_{\text{outside function}} \cdot \underbrace{g'(x)}_{\text{inside function}}$.

Alternatively, g is a function of x.
& f is a function of $g(x)$, say u.

$$\text{Then, } \frac{d}{dx}[h(x)] = \frac{d}{dx}[f(u)]$$
$$= \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$$
$$= f'(u) \cdot [u']$$
$$= f'(u) \cdot g'(x).$$

Examples

1. $h(x) = [g(x)]^n$

$$h'(x) = n[g(x)]^{n-1} \cdot g'(x).$$

$$h(x) = (\sin x)^n \Rightarrow h'(x) = n \sin^{n-1} x \cdot \cos x.$$

2. $h(x) = g(mx)$

$$h'(x) = g'(mx) \cdot [mx]' = g'(mx) \cdot m$$

$$h(x) = \sin 3x \Rightarrow h'(x) = \cos(3x) \cdot 3$$

Try: $h(x) = \cos(7x^2)$